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Maximum number of r-edge-colorings such that all copies of $\mathbf{K_k}$ are rainbow

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Abstract

We consider a version of the Erdős-Rothschild problem for families of graph patterns. For any fixed $k \geq 3$, let $r_0(k)$ be the largest integer such that the following holds for all $2 \leq r \leq r_0(k)$ and all sufficiently large n: The Turán graph $T_{k-1}(n)$ is the unique n-vertex graph G with the maximum number of r-edge-colorings such that the edge set of any copy of K_k in G is rainbow. We use the regularity lemma of Szemerédi and linear programming to obtain a lower bound on the value of $r_0(k)$. For a more general family $\mathcal{P}$ of patterns of K_k, we also prove that, in order to show that the Turán graph $T_{k-1}(n)$ maximizes the number of $\mathcal{P}$-free r-edge-colorings over n-vertex graphs, it suffices to prove a related stability result.

Keywords: Edge-colorings; Turán Problem; Erdős-Rothschild Problem

1. Introduction

For a fixed graph F and a fixed positive integer n, the classical *Turán problem* asks for the maximum number $\mathrm{ex}(n, F)$ of edges in an F-*free* n-vertex graph, that is, in an n-vertex graph G with no copy of F as a subgraph.

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Moreover, it asks for a characterization of the F-free n-vertex graphs G with $\mathrm{ex}(n, F)$ edges, which are said to be *F-extremal*. This problem was fully solved by Turán [13] when $F = K_k$ is a complete graph on $k \geq 3$ vertices. He showed that any K_k-extremal n-vertex graph is isomorphic to the *Turán graph* $T_{k-1}(n)$, namely the complete $(k-1)$-partite graph whose parts have size as equal as possible. There are many graphs F for which the Turán problem has not yet been solved, particularly when F is bipartite. In addition to its direct influence in Extremal Combinatorics, the Turán problem has given rise to a large number of related problems. In this paper, we investigate one such problem involving edge-colorings, known as the *Erdös-Rothschild problem* [6].

Let $[r] = \{1, \ldots, r\}$ be a set of colors. For simplicity, we write r-coloring to refer to an r-edge-coloring. For a fixed graph F, a *pattern* P of F is a partition of its edge set. An r-coloring of a graph G is said to be *(F, P)-free* if G does not contain a copy of F in which the partition of the edge set induced by the coloring is isomorphic to P. Let $\mathcal{P}$ be a pattern family, i.e., a set of pairs (F, P) for which F is a graph and P is a pattern of F. If all patterns in the family are of the same graph F, we refer to a *pattern family* of F. We say that an r-colored graph $\widehat{G}$ is $\mathcal{P}$-free if $\widehat{G}$ is (F, P)-free for all pairs $(F, P) \in \mathcal{P}$.

In 1974, Erdős [6] wrote that, together with Rothschild, he considered a question about r-colorings of a graph G that avoid monochromatic copies of a given graph F. In our notation, they considered $\mathcal{K}_k^M$-free r-colorings of G for the family $\mathcal{K}_k^M = \{(K_k, \{E(K_k)\})\}$ consisting of the pattern of K_k where all edges lie in the same class of the partition. Erdős and Rothschild conjectured that, for any fixed $k \geq 3$, the Turán graph $T_{k-1}(n)$ admits the largest number of $\mathcal{K}_k^M$-free 2-colorings among all n-vertex graphs, where n is large. Yuster [14] showed that this is true for $k = 3$ and $n \geq 6$, while Alon, Balogh, Keevash and Sudakov [1] proved the remaining cases for large n (see also [8]). For larger values of r, Erdős and Rothschild also asked whether the following holds for all or almost all choices of graph F: For all fixed $\varepsilon > 0$, does any large n-vertex graph G admit at most $r^{(1+\varepsilon)\,\mathrm{ex}(n,F)}$ distinct r-colorings with no monochromatic copy of F? The work of Alon, Balogh, Keevash and Sudakov [1] also implies that the answer to this question is no for $r \geq 4$ and all graphs with (vertex) chromatic number at least three.

More generally, this problem may be stated as follows. Given a positive integer r, a pattern family $\mathcal{P}$ and a graph G, let $C_{r,\mathcal{P}}(G)$ be the set of all $\mathcal{P}$-free r-colorings of a graph G. We write $c_{r,\mathcal{P}}(G) = |C_{r,\mathcal{P}}(G)|$ and

$$c_{r,\mathcal{P}}(n) = \max\left\{c_{r,\mathcal{P}}(G) \colon |V(G)| = n\right\}. \tag{1}$$

We say that an n-vertex graph G is *$(r, \mathcal{P})$-extremal* if $c_{r,\mathcal{P}}(n) = c_{r,\mathcal{P}}(G)$. Clearly, determining $c_{r,\mathcal{P}}(n)$ and the corresponding set of $(r, \mathcal{P})$-extremal graphs is a generalized version of the problem considered by Erdős and Rothschild, where $\mathcal{P}$ consisted of the monochromatic pattern of a given graph. Note that the Turán problem may also be formulated in this setting: for a graph F, consider the pattern family $\mathcal{P}_F^{all}$ that contains all possible patterns of F and fix $r \geq 2$. Clearly, a graph G admits a $\mathcal{P}_F^{all}$-free coloring if and only if it is F-free, in which case $c_{r,\mathcal{P}_F^{all}}(G) = r^{|E(G)|}$. Thus G is $(r, \mathcal{P}_F^{all})$-extremal if and only if it is F-extremal.

Balogh [3] was the first to study a problem of this type for non-monochromatic colorings. Recently, there have been several results in this direction, both in the monochromatic and in more general settings. Another class of patterns that has been studied is the class of *rainbow* patterns. Given a graph F, the rainbow pattern F^R is the partition of $E(F)$ such that each class contains a single edge. For any fixed $k \geq 3$ and $r \geq \binom{k+1}{2}^{8k+4}$, Hoppen, Lefmann and Odermann [11] showed that, for large n, the Turán graph $T_{k-1}(n)$ is the unique n-vertex $(r, \mathcal{K}_k^R)$-extremal graph, where $\mathcal{K}_k^R = \{(K_k, K_k^R)\}$. We also refer to [4, 5] for recent results involving rainbow patterns. A simple consequence of known results is that, if $\mathcal{P}_k^*$ is any pattern family of K_k that contains K_k^R, then there exists r_1 such that, for all $r \geq r_1$, the Turán graph $T_{k-1}(n)$ is the unique n-vertex $(r, \mathcal{P}_k^*)$-extremal graph for large n. On the other hand, if $\mathcal{P}_k^*$ is any pattern family of K_k that does not contain K_k^R, there exists r_2 such that, for all $r \geq r_2$, the Turán graph $T_{k-1}(n)$ is not $(r, \mathcal{P}_k^*)$-extremal for large n.

In this paper, we consider the family $\mathcal{P}_k$ of all patterns of K_k that have fewer than $\binom{k}{2}$ classes, that is, all the *non-rainbow patterns* of K_k. This means that colorings in $C_{r,\mathcal{P}_k}(G)$ are such that all copies of K_k are rainbow. Clearly, for $2 \leq r < \binom{k}{2}$ colors the Turán graph $T_{k-1}(n)$ is optimal and it is the only such n-vertex graph for any n. The discussion in the previous paragraph implies that there is a largest positive integer $r_0(k)$ such that, for all $2 \leq r \leq r_0(k)$ and all sufficiently large n, the Turán graph $T_{k-1}(n)$ is the unique n-vertex $(r, \mathcal{P}_k)$-extremal graph, see Section 5 for more information. Our main result is the following lower bound on $r_0(k)$ for all $k \geq 3$. For $k = 3$, it leads to $12 \leq r_0(3) \leq 26$, recovering a result of [9].

Theorem 1.1. *For a fixed integer $k \geq 3$, consider the quantity $r_0^*(k)$ defined in (13). For all integers $2 \leq r \leq r_0^*(k)$, there is $n_0 > 0$ such that any graph G of order $n > n_0$ satisfies $c_{r,\mathcal{P}_k}(G) \leq r^{\mathrm{ex}(n,K_k)}$, and equality holds if and only if G is isomorphic to the Turán graph $T_{k-1}(n)$.*

The proof implies that $r_0^*(k) \geq 2\binom{k}{2}$ for all $k \geq 3$. To prove Theorem 1.1, we use a stability approach, in which we first show that all graphs that admit a large number of feasible colorings must be structurally similar to the Turán graph. This may be formalized as follows.

Definition 1.2. *Let F be a graph with chromatic number $\chi(F) = k \geq 3$ and let $\mathcal{P}_F$ be a pattern family of F. We say that $\mathcal{P}_F$ satisfies the* Color Stability Property *for a positive integer r if, for every $\delta > 0$ there exists n_0 with the following property. If $n > n_0$ and G is an n-vertex graph such that $c_{r,\mathcal{P}_F}(G) \geq r^{\mathrm{ex}(n,F)}$, then there exists a partition $V(G) = V_1 \cup \cdots \cup V_{k-1}$ such that $\sum_{i=1}^{k-1} e(V_i) < \delta n^2$, where $e(V_i)$ denotes the number of edges of G with both ends in V_i.*

Hoppen, Lefmann and Odermann [11] showed that if the family $\mathcal{P} = \{(K_k, P)\}$ satisfies the Color Stability Property for a positive integer r and for a pattern P for which there exists a vertex v in K_k such that all incident edges are in different classes of P then the Turán graph $T_{k-1}(n)$ is $(r,\mathcal{P})$-extremal for large n. To derive their main result, it was sufficient to prove that $\mathcal{K}_k^R$ satisfies the Color Stability Property for $r \geq r_0$.

To prove Theorem 1.1, we obtain a similar result for pattern families. For positive integers r and k, we say that a pattern family $\mathcal{P}_k^S$ of K_k is $(r, k-1)$-*vertex saturated* if, for any pattern P' of the star $K_{1,k-1}$ with $(k-1)$ edges, there exists a pattern $P \in \mathcal{P}_k^S$ and a vertex $v \in V(K_k)$ such that the pattern induced by P on the edges incident with v is isomorphic to P'. We prove the following.

Theorem 1.3. *For fixed integers $k \geq 3$ and $r \geq 2$, let $\mathcal{P}_k^S$ be a $(r, k-1)$-vertex saturated pattern family of K_k. If $\mathcal{P}_k^S$ satisfies the Color Stability Property for r, then there exists an integer $n_0 > 0$ such that the following holds for every $n > n_0$. If G is an n-vertex graph, then $c_{r,\mathcal{P}_k^S}(G) \leq r^{\mathrm{ex}(n,K_k)}$. Moreover, the only graph on n vertices for which the number of $(r,\mathcal{P}_k^S)$-free r-colorings is equal to $r^{\mathrm{ex}(n,K_k)}$ is the Turán graph $T_{k-1}(n)$.*

Using Theorem 1.3, we may prove Theorem 1.1 by showing that the pattern family $\mathcal{P}_k$ satisfies the Color Stability Property for $2 \leq r \leq r_0^*(k)$.

Lemma 1.4. *For a fixed integer $k \geq 3$ and the quantity $r_0^*(k)$ defined in (13), the following holds for all integers $2 \leq r \leq r_0^*(k)$. For every $\delta > 0$ there exists an n_0 such that if $G = (V,E)$ is a graph on $n \geq n_0$ vertices such that $c_{r,\mathcal{P}_k}(G) \geq r^{\mathrm{ex}(n,K_k)}$, then there is a partition $V = V_1 \cup \cdots \cup V_{k-1}$ of its vertex set such that $\sum_{i=1}^{k-1} e(V_i) \leq \delta n^2$.*

The proof of our results combines the general strategy in [1] and [11] with linear programming. In the next section, we introduce the basic preliminary results and notation required. In Section 3, we prove Theorem 1.3. This leads to our main result in combination with Lemma 1.4, proved in Section 4. It turns out that, for any fixed integer $k \geq 3$, there is a largest positive integer $r_0(k)$ for which the statements of Theorem 1.1 and Lemma 1.4, respectively, are true. Upper bounds on $r_0(k)$ are discussed in Section 5.

2. Notation and Basic Tools

Let $G = (V,E)$ be a graph. We write $e(G) = |E(G)|$ for the number of edges in G. For subsets $A, B \subseteq V$ of vertices we write $e(A) = e(A,A)$ for the number of edges with both ends in A and $e(A,B)$ for the number of edges with one end in A and other in B. Let $d(A,B) = e(A,B)/(|A||B|)$ be the edge-density between A and B.

One important tool that we use is Szemerédi's regularity lemma. For $\varepsilon > 0$ the pair $\{A, B\}$ is called ε-*regular* if, for every $X \subseteq A$ and $Y \subseteq B$ satisfying $|X| > \varepsilon|A|$ and $|Y| > \varepsilon|B|$, we have $|d(X,Y) - d(A,B)| < \varepsilon$. An *equitable partition* of a set V is a partition of V into pairwise disjoint classes $V_1, \ldots, V_m$ such that $||V_i| - |V_j|| \leq 1$ for all $i, j \in [m]$. An equitable partition $V_1 \cup \cdots \cup V_m$ of the vertex set V of G is called ε-*regular* if at most $\varepsilon\binom{m}{2}$ of the pairs $\{V_i, V_j\}$ are not ε-regular. We use the following version of the regularity lemma for edge-colored graphs, which may be found in [12, Theorem 1.18].

Lemma 2.1. *For every $\varepsilon > 0$ and every positive integer r, there exists an $M = M(\varepsilon, r)$ such that the following holds. If the edges of a graph G of order $n > M$ are r-colored $E(G) = E_1 \cup \cdots \cup E_r$, then there is a partition of the vertex set*

$V(G) = V_1 \cup \cdots \cup V_m$, with $1/\varepsilon \le m \le M$, which is ε-regular simultaneously with respect to the graphs $G_i = (V, E_i)$ for all $i \in [r]$.

A partition $V_1 \cup \cdots \cup V_m$ of $V(G)$ as in Lemma 2.1 will be called a *multicolored ε-regular partition*. For $\eta > 0$, we may define a *multicolored cluster graph* $\mathcal{H}(\eta)$ associated with this partition, where the vertex set is $[m]$ and $e = \{i, j\}$ is an edge of $\mathcal{H}(\eta)$ if $\{V_i, V_j\}$ is a regular pair in G *for every* color $c \in [r]$ and the edge-density of the pair $\{V_i, V_j\}$ is at least η for some color $c \in [r]$. Each edge e is assigned the list L_e containing all colors for which its edge-density is at least η, so that $|L_e| \ge 1$ for all $e \in E(\mathcal{H}(\eta))$. We say that a multicolored cluster graph $\mathcal{H}$ contains a colored graph $\widehat{F}$, if $\mathcal{H}$ contains a copy of the (uncolored) underlying graph of $\widehat{F}$ for which the color of each edge of $\widehat{F}$ is contained in the list of the corresponding edge in $\mathcal{H}$. More generally, if F is a graph with color pattern P, we say that $\mathcal{H}$ contains (F, P) if it contains some colored copy of F with pattern P. In connection with this definition, we may obtain the following embedding result, whose proof follows from arguments such as in the proof of [12, Theorem 2.1].

Lemma 2.2. *For every $\eta > 0$ and all positive integers k and r, there exists $\varepsilon = \varepsilon(r, \eta, k) > 0$ and a positive integer $n_0(r, \eta, k)$ with the following property. Suppose that $\widehat{G}$ is an r-colored graph on $n > n_0$ vertices with a multicolored ε-regular partition $V = V_1 \cup \cdots \cup V_m$ which defines the multicolored cluster graph $\mathcal{H} = \mathcal{H}(\eta)$. Let F be a fixed k-vertex graph with a color pattern P. If $\mathcal{H}$ contains (F, P), then the graph $\widehat{G}$ contains (F, P).* □

For the proof of Theorem 1.3, we are going to use the following auxiliary proposition.

Proposition 2.3. *Let $r > 0$ and $k \ge 2$ be positive integers and let $\widehat{K_k}$ be an r-colored copy of the complete graph K_k with vertex set $\{v_1, \ldots, v_k\}$ such that each edge $\{v_i, v_j\}$ has color $\alpha_{i,j} \in [r]$. Let $\omega\colon [k] \to (0, 1]$, with $\omega(i) \le 1/(i-1)$ for all $1 < i \le k$, be a non-increasing function and fix $\beta \ge \omega(i)/\omega(i-1)$ for all $1 < i \le k$. Let $\widehat{G}$ be a colored graph whose vertex set contains mutually disjoint sets $W_1, \ldots, W_k$ with the following property. If for every pair $\{i, j\} \subseteq [k]$ and all subsets $X_i \subseteq W_i$, where $|X_i| \ge \omega(k)|W_i|$, and $X_j \subseteq W_j$, where $|X_j| \ge \omega(k)|W_j|$, there are at least $\beta|X_i||X_j|$ edges of color $\alpha_{i,j}$ between X_i and X_j in $\widehat{G}$, then $\widehat{G}$ contains a copy of $\widehat{K_k}$ with one vertex in each set W_i.*

Another basic tool in our paper is a recent stability result for graphs due to Füredi [7].

Theorem 2.4. *Let $G = (V, E)$ be a K_k-free graph on m vertices. If $|E| = \mathrm{ex}(m, K_k) - t$, then there exists a partition $V = V_1 \cup \ldots \cup V_{k-1}$ with $\sum_{i=1}^{k-1} e(V_i) \le t$.*

We also use the following lemma due to Alon and Yuster [2, Lemma 2.3].

Lemma 2.5. *Let $0 < t \le m^2/(4(k-1)^2)$ be fixed and let G be a $(k-1)$-partite graph on m vertices with partition $V(G) = U_1 \cup \cdots \cup U_{k-1}$ and with at least $\mathrm{ex}(m, K_k) - t$ edges. If we add at least $(2k-1)t$ new edges to G, then in the resulting graph there is a copy of K_k with exactly one new edge, both of whose endpoints lie in U_i, for some $i \in [k-1]$.*

We use the following fact about the sizes of the classes in a $(k-1)$-partite graph with many edges.

Proposition 2.6. *[11, Proposition 2.7] Let $G = (V, E)$ be a $(k-1)$-partite graph on m vertices with $(k-1)$-partition $V = V_1 \cup \cdots \cup V_{k-1}$. If, for some $t \ge (k-1)^2$, the graph G contains at least $\mathrm{ex}(m, K_k) - t$ edges, then for $i \in \{1, \ldots, k-1\}$ we have $||V_i| - m/(k-1)| < \sqrt{2t}$.*

Let $H\colon [0, 1] \to [0, 1]$ be the *entropy function*, where $H(x) = -x \log_2 x - (1-x) \log_2(1-x)$ with $H(0) = H(1) = 0$. It will be convenient to use the following inequalities for all $0 \le \alpha \le 1$ and all $0 \le x \le 1/8$:

$$\binom{n}{\alpha n} \le 2^{H(\alpha)n} \quad \text{and} \quad H(x) \le -2x \log_2 x. \tag{2}$$

3. Proof of Theorem 1.3

For the proof of Theorem 1.3 we are going to use the following result by Hoppen, Lefmann and Nolibos [10].

Theorem 3.1. *Let $\mathcal{P}$ be a pattern family of a complete graph K_k, with $k \ge 3$, and let $r \ge 2$ be an integer. For any positive integer n, there exists an n-vertex complete multipartite graph that is $(r, \mathcal{P})$-extremal. Moreover, if there is an n-vertex graph G that is $(r, \mathcal{P})$-extremal and G is not a complete multipartite graph, then there are at least two non-isomorphic n-vertex complete multipartite graphs that are $(r, \mathcal{P})$-extremal.*

Proof of Theorem 1.3. For $k \geq 3$ and $r \geq 2$, let $\mathcal{P}_k^S$ be an $(r, k-1)$-vertex saturated pattern family of K_k that satisfies the Color Stability Property (Definition 1.2) for r. Let n_0 be given by the Color Stability Property for $\delta = 1/(5^{2k+1} r^{4k} k^2)$.

By Theorem 3.1, let G be a complete n-vertex s-partite graph, with $n > n_0$, such that $c_{r,\mathcal{P}_k^S}(G) \geq r^{\mathrm{ex}(n,K_k)}$. By Turán's Theorem, G is isomorphic to $T_{k-1}(n)$ or $s \geq k$. For a contradiction, assume that $s \geq k$. Let $V(G) = V_1 \cup \cdots \cup V_{k-1}$ be a partition of the vertex set of G such that $\sum_{i=1}^{k-1} e(V_i)$ is minimum. This implies that, if $v \in V_i$ and $j \in [k-1]$, we have $|N(v) \cap V_j| \geq |N(v) \cap V_i|$. Moreover, by the Color Stability Property, we have $\sum_{i=1}^{k-1} e(V_i) \leq \delta n^2$. Thus, by Proposition 2.6, for all $i \in [k-1]$ we have $|V_i| \geq n/(k-1) - \sqrt{2\delta}n$.

Since G is a complete s-partite graph and $s \geq k$, there is an edge uv such that u and v lie in the same class, say u and v are in V_1; moreover, every vertex $x \in V(G)$ must be adjacent to either u or v. So, if we assume that $|N(v) \cap V_1| \geq |N(u) \cap V_1|$, we have, by the choice of δ, that $|N(v) \cap V_i| \geq n/(2k)$ for all $i \in [k-1]$.

We shall prove that G must have fewer than $r^{\mathrm{ex}(n,K_k)}$ distinct $\mathcal{P}_k^S$-free colorings, the desired contradiction. To this end, let us analyse the structure of a fixed $\mathcal{P}_k^S$-free r-coloring $\widehat{G}$ of G. For $i \in [k-1]$, define $W_i = N(v) \cap V_i$. By the pigeonhole principle, there exist colors $\alpha_1, \ldots, \alpha_{k-1} \in [r]$, not necessarily distinct, and subsets $W_1^{\alpha_1}, \ldots, W_{k-1}^{\alpha_{k-1}}$, with $W_i^{\alpha_i} \subseteq W_i$, such that all edges between v and $W_i^{\alpha_i}$ have color α_i and $|W_i^{\alpha_i}| \geq n/(2rk)$.

By definition of $\mathcal{P}_k^S$, there exists a pattern P such that $(K_k, P) \in \mathcal{P}_k^S$ and a vertex $x \in V(K_k)$ such that the pattern of the edges incident with x is isomorphic to the pattern induced by the colors $\alpha_1, \ldots, \alpha_{k-1}$. Since $\widehat{G}$ is (K_k, P)-free, we apply Proposition 2.3 with $\beta = 1/5r^2$ and $\omega(i) = 1/(5r^2)^i$ for every $i \in [k]$ to fix a color $h \in [r]$ and a pair (X_i, X_j) that satisfies $X_i \subseteq W_i^{\alpha_i}$, where $|X_i| \geq \omega(k-1)|W_i^{\alpha_i}|$ and $X_j \subseteq W_j^{\alpha_j}$, where $|X_j| \geq \omega(k-1)|W_j^{\alpha_i}|$ and there are fewer than $\beta|X_i||X_j|$ edges of color h between $|X_i|$ and $|X_j|$.

We use this to bound the size of $C_{r,\mathcal{P}_k^S}(G)$. Note that there are r choices for the color h and at most 2^{2n} choices for the sets X_i and X_j. Once we fixed the color h and the sets X_i and X_j we have at most

$$\binom{|X_i||X_j|}{|X_i||X_j|/5r^2} \cdot (r-1)^{|X_i||X_j|} \overset{(2)}{\leq} 2^{H(1/5r^2)|X_i||X_j|} \cdot (r-1)^{|X_i||X_j|} \overset{(2)}{\leq} \left((5r^2)^{\frac{2}{5r^2}} \cdot (r-1)\right)^{|X_i||X_j|}$$

ways to color the edges between X_i and X_j. We have at most $\mathrm{ex}(n, K_k) + \delta n^2 - |X_i||X_j|$ other edges in G. Finally, since $\delta = \frac{1}{5^{2k+1} r^{4k} k^2}$ and $|X_i||X_j| \geq \frac{n^2}{4r^2k^2(5r^2)^{2k-2}} > \frac{n^2}{5^{2k-1} r^{2(2k-1)} k^2}$, we have

$$\begin{aligned}
|C_{r,\mathcal{P}_k^S}(G)| &\leq r2^{2n} \cdot (5r^2)^{\frac{2}{5r^2}|X_i||X_j|} \cdot (r-1)^{|X_i||X_j|} \cdot r^{\mathrm{ex}(n,K_k)+\delta n^2 - |X_i||X_j|} \\
&\leq \left(25r^4 \left(\frac{r-1}{r}\right)^{5r^2}\right)^{|X_i||X_j|/5r^2} \cdot r^{\mathrm{ex}(n,K_k)+2\delta n^2} \\
&< r^{\mathrm{ex}(n,K_k)+2\delta n^2 - |X_i||X_j|/5r^2} < r^{\mathrm{ex}(n,K_k)},
\end{aligned}$$

which leads to the desired contradiction. □

4. Color Stability Property of the family $\mathcal{P}_k$

The aim of this section is to prove Lemma 1.4. In combination with Theorem 1.3, it establishes Theorem 1.1.

Proof of Lemma 1.4. Let $k \geq 3$ and $\delta > 0$. In this proof, it is convenient to write $\ell = \ell(k) = \binom{k}{2} - 1$. Consider the quantities $Y_{r,k}^* < r$ and $r_0^*(k)$ to be defined in (13) through the solution of the linear programs in (12). Fix an integer r such that $2 \leq r \leq r_0^*(k)$. With foresight, we consider auxiliary constants $\xi > 0$ and $\eta > 0$ such that

$$\xi < \frac{1}{8(k-1)(k-2)}, \quad \xi < \frac{\delta}{8k-2}, \quad r^{r\eta + H(r\eta)} < \left(\frac{r}{Y_{r,k}^*}\right)^{\xi} \text{ and } \eta < \frac{\delta}{2r}. \tag{3}$$

Let $\varepsilon = \varepsilon(r, \eta, k) > 0$ satisfy the assumption in Lemma 2.2, and assume w.l.o.g. that $\varepsilon < \eta/2$. Fix $M = M(r, \varepsilon)$ given by Lemma 2.1. Let G be an n-vertex graph, where n is large, that satisfies the statement of Lemma 1.4.

By Lemma 2.1, any colored graph $\widehat{G}$ given by a $\mathcal{P}_k$-free r-coloring of $G = (V, E)$ is associated with a multicolored ε-regular partition $V = V_1 \cup \cdots \cup V_m$, which leads to a multicolored cluster graph $\mathcal{H} = \mathcal{H}(\eta)$, where $1/\varepsilon \leq m \leq M$. The main step in our proof is showing that there is one such $\mathcal{H}$ with a large number of edges with large lists.

Claim 4.1. *There exists a multicolored cluster graph $\mathcal{H} = \mathcal{H}(\eta)$ such that*

$$e_{\lceil r/\ell \rceil}(\mathcal{H}) + \cdots + e_r(\mathcal{H}) \geq \mathrm{ex}(m, K_k) - \xi m^2. \tag{4}$$

Before proving this claim, we show that it implies the desired result. Fix a multicolored cluster graph $\mathcal{H}(\eta)$ that satisfies (4). Using Theorem 2.4, let $\widehat{\mathcal{H}}$ be a $(k-1)$-partite subgraph of $\mathcal{H}$ with $V(\widehat{\mathcal{H}}) = U_1 \cup \cdots \cup U_{k-1}$ and $e(\widehat{\mathcal{H}}) \geq \mathrm{ex}(m, K_k) - 2\xi m^2$. We claim that $e_1(\mathcal{H}) + \cdots + e_{\lceil r/\ell \rceil - 1}(\mathcal{H}) \leq (4k-2)\xi m^2$. Otherwise, Lemma 2.5 can be applied and the graph obtained by adding the edges $E_1 \cup \cdots \cup E_{\lceil r/\ell \rceil - 1}$ to $\widehat{\mathcal{H}}$ would contain a copy of K_k such that at most one of the edges is in $E_1 \cup \cdots \cup E_{\lceil r/\ell \rceil - 1}$ and all the others are in $E_{\lceil r/\ell \rceil} \cup \cdots \cup E_r$. The sum of the sizes of the lists of edges in this copy is at least $1 + \lceil r/\ell \rceil \cdot \ell > r$, so that some color appears in distinct lists. As a consequence, $\widehat{\mathcal{H}}$ contains (K_k, P) for some $P \in \mathcal{P}_k$, so that, by Lemma 2.2, the same happens for any coloring $\hat{G}$ leading to $\mathcal{H}$, a contradiction. As a consequence, the number of edges of $\mathcal{H}$ with both ends in a same class U_i is at most $(4k-1)\xi m^2$.

Let $W_i = \cup_{j \in U_i} V_j$ for $i \in \{1, \ldots, k-1\}$. Then, by our choice of η and ξ, we have

$$e_G(W_1) + \cdots + e_G(W_{k-1}) \leq r\eta n^2 + (n/m)^2 \cdot (e_{\mathcal{H}}(U_1) + \cdots + e_{\mathcal{H}}(U_{k-1})) < \delta n^2,$$

as required. This proves Lemma 1.4 assuming the validity of Claim 4.1. □

Proof of Claim 4.1. Suppose for a contradiction that all multicolored cluster graphs $\mathcal{H}(\eta)$ satisfy

$$e_{\lceil r/\ell \rceil}(\mathcal{H}) + \cdots + e_r(\mathcal{H}) < \mathrm{ex}(m, K_k) - \xi m^2. \tag{5}$$

To reach the contradiction, we shall estimate the number of r-colorings of G that are associated with each regular partition $V(G) = V_1 \cup \cdots \cup V_m$ and each multicolored cluster graph $\mathcal{H} = \mathcal{H}(\eta)$ associated with it. First, notice that for any multicolored ε-regular partition of G we have at most $r\eta n^2$ edges e that satisfy one of the following conditions: (a) the endpoints of e are in the same class; (b) e connects vertices in an irregular pair; (c) e connects vertices in two classes for which the color of e has density less than η. This set of edges can be colored in at most $r^{r\eta n^2}$ different ways. Note that all edges vw of G that have not been considered so far must be such that $v \in V_i$, $w \in V_j$ and $e = ij \in E(\mathcal{H})$, for some $i, j \in [m]$, $i \neq j$. Also, the color assigned to it is in L_e. Let $E_j(\mathcal{H}) = \{e \in E(\mathcal{H}) : |L_e| = j\}$ and $e_j(\mathcal{H}) = |E_j(\mathcal{H})|$, for all $j \in [r]$. The number of r-colorings of G that give rise to the partition $V = V_1 \cup \cdots \cup V_m$ and to the multicolored cluster graph $\mathcal{H}$ is bounded above by

$$\binom{n^2}{r\eta n^2} \cdot r^{r\eta n^2} \cdot \left(\prod_{j=1}^{r} j^{e_j(\mathcal{H})} \right)^{\left(\frac{n}{m}\right)^2} \leq 2^{H(r\eta)n^2} \cdot r^{r\eta n^2} \cdot \left(\prod_{j=1}^{r} j^{e_j(\mathcal{H})} \right)^{\left(\frac{n}{m}\right)^2}. \tag{6}$$

Let $\mathcal{H}^*$ be a multicolored cluster graph on m vertices that maximizes (6). There are at most M^n partitions on $m \leq M$ classes and for each of these partitions there are at most $2^{rM^2/2}$ choices for the multicolored cluster graph. Thus, using (6), summing over all partitions and all corresponding multicolored cluster graphs $\mathcal{H}$, and assuming that n is large, the number of $\mathcal{P}_k$-free r-colorings of G is bounded above by

$$M^n \cdot 2^{H(r\eta)n^2} \cdot r^{r\eta n^2} \cdot 2^{\frac{rM^2}{2}} \cdot \left(\prod_{j=1}^{r} j^{e_j(\mathcal{H}^*)} \right)^{\left(\frac{n}{m}\right)^2} \leq r^{(r\eta + H(r\eta))n^2} \cdot \left(\prod_{j=1}^{r} j^{e_j(\mathcal{H}^*)} \right)^{\left(\frac{n}{m}\right)^2}. \tag{7}$$

First, suppose that $2 \leq r \leq 2\ell$. By (5), (7) and our choice of ξ, we have for n large that

$$r^{(r\eta + H(r\eta))n^2} \cdot r^{\mathrm{ex}(n,K_k) - \xi n^2} \overset{(3)}{\ll} r^{\mathrm{ex}(n,K_k)}, \tag{8}$$

which is a contradiction. This means that we may always fix $r_0^*(k) \geq 2\ell$.

Assume next that $r > 2\ell$. For a fixed positive integer $y \leq \ell$, let

$$z(y) = \min \left\{ z \in \mathbb{N} \colon \left(\binom{k}{2} - y \right) z > r - \binom{k}{2} \right\}. \tag{9}$$

Note that $z(y) + 1 \geq \lceil r/\ell \rceil$ for all $1 \leq y \leq \ell$. Given a y-element set $S \subset [r]$ and $j \in \{1, \ldots, z(y)\}$, let $E_j(S, int_{\geq 1}; \mathcal{H})$ be the set of all edges $e' \in E_j(\mathcal{H})$ that satisfy $|L_{e'} \cap S| \geq 1$, and let $e_j(S, int_{\geq 1}; \mathcal{H}) = |E_j(S, int_{\geq 1}; \mathcal{H})|$. Counting arguments give the following facts.

Proposition 4.2. *Consider a $\mathcal{P}_k$-free multicolored cluster graph $\mathcal{H}$ and let $1 \le y \le \ell$.*

(a) For every y-element subset $S \subset [r]$, the subgraph $\mathcal{H}'$ of the multicolored cluster graph $\mathcal{H}$ with edge set $\bigcup_{j=1}^{z(y)} E_j(S, int_{\ge 1}; \mathcal{H}) \cup \bigcup_{p=z(y)+1}^{r} E_p(\mathcal{H})$ is K_k-free.

(b) There exists a y-element subset $S \subset [r]$ such that $\left|\bigcup_{j=1}^{z(y)} E_j(S, int_{\ge 1}; \mathcal{H})\right| \ge \sum_{j=1}^{z(y)} \frac{\binom{r}{y} - \binom{r-j}{y}}{\binom{r}{y}} \cdot |E_j(\mathcal{H})|$.

Proposition 4.2 leads to the following inequality:

$$\sum_{j=1}^{z(y)} \frac{\binom{r}{y} - \binom{r-j}{y}}{\binom{r}{y}} \cdot e_j(\mathcal{H}) + \sum_{p=z(y)+1}^{r} e_p(\mathcal{H}) \le \mathrm{ex}(m, K_k). \tag{10}$$

Proposition 4.3. *Consider a of $\mathcal{P}_k$-free multicolored cluster graph $\mathcal{H}$. For $q = 1, \dots, \lfloor r/\binom{k}{2} \rfloor$, let $\mathcal{H}'_q$ be the subgraph of $\mathcal{H}$ with vertex set $[m]$ and edge set $E_q \cup \cdots \cup E_{r-\ell\cdot q}$, and fix a $(k-1)$-partite subgraph B'_q of $\mathcal{H}'_q$.*

(a) The subgraph $\mathcal{H}''_q$ of the multicolored cluster graph $\mathcal{H}$ determined by the edge set $E(B'_q) \cup \bigcup_{p=r-\ell\cdot q+1}^{r} E_p(\mathcal{H})$ is K_k-free.

(b) Moreover, there exists a $(k-1)$-partite subgraph B'_q of $\mathcal{H}'_q$ such that $|E(B'_q)| \ge \frac{k-2}{k-1} \cdot |E(\mathcal{H}'_q)|$.

For $q = 1, \dots, \lfloor r/\binom{k}{2} \rfloor$, Proposition 4.3 leads to the following inequality:

$$\frac{k-2}{k-1} \cdot \sum_{j=q}^{r-\ell\cdot q} e_j(\mathcal{H}) + \sum_{p=r-\ell\cdot q+1}^{r} e_p(\mathcal{H}) \le \mathrm{ex}(m, K_k). \tag{11}$$

Finding the maximum of $\prod_{j=1}^{r} j^{e_j(\mathcal{H}^*)}$ of (7) is equivalent to maximizing $e_2 \ln 2 + e_3 \ln 3 + \cdots + e_r \ln r$, which is a linear objective function with respect to the variables $e_2, \dots, e_r \ge 0$. Together with the linear constraints of (10) and (11), we obtain a linear program as follows. Given $\mathcal{H}$, fix $y \le \ell$ and set $\zeta(\mathcal{H}) = \left(\mathrm{ex}(m, K_k) - e_{z(y)+1}(\mathcal{H}) - \cdots - e_r(\mathcal{H})\right)/m^2$, so that $\zeta(\mathcal{H}) > \xi$ by (5). This leads to the linear program

$$\begin{aligned}
\max \quad & x_2 \ln 2 + \cdots + x_{z(y)} \ln z(y) \\
\text{s.t.} \quad & \sum_{j=2}^{z(y')} \frac{\binom{r}{y'} - \binom{r-j}{y'}}{\binom{r}{y'}} \cdot x_j + \sum_{j=z(y')+1}^{z(y)} x_j \le 1, \quad y' = 1, \dots, y \\
& \frac{k-2}{k-1} \cdot \sum_{j=\max\{2,q\}}^{r-\ell\cdot q} x_j + \sum_{p=r-\ell\cdot q+1}^{z(y)} x_p \le 1, \quad q = \left\lceil \frac{r - z(y)}{\ell} \right\rceil, \dots, \left\lfloor \frac{r}{\binom{k}{2}} \right\rfloor \\
& x_2, \dots, x_{z(y)} \ge 0,
\end{aligned} \tag{12}$$

where x_i plays the role of $e_i(\mathcal{H})/(\zeta(\mathcal{H})m^2)$. We set $Y(r, k, y) = e^{u(r,k,y)}$ where $u(r, k, y)$ is the optimal value of the linear program in (12). We are now ready to define our lower bound $r_0^*(k)$ on $r_0(k)$. Note that it does not depend on any aspect of the proof other than the parameters r and k. For $r > 2\ell$, set

$$Y^*_{r,k} = \min\{Y(r, k, y) \colon 1 \le y \le \ell\}, \; r_0^*(k) = \min\{r \in \mathbb{N} \colon Y^*_{r+1,k} \ge r + 1\}. \tag{13}$$

Let y^* be a value of y that achieves the minimum in the first equation of (13). With the above constraints, for any multicolored cluster graph $\mathcal{H}$, we have

$$\begin{aligned}
\prod_{j=2}^{r} j^{e_j(\mathcal{H})} &= \left(\prod_{j=2}^{z(y^*)} j^{e_j(\mathcal{H})}\right) \cdot \left(\prod_{j=z(y^*)+1}^{r} j^{e_j(\mathcal{H})}\right) \le \left(\prod_{j=2}^{z(y^*)} j^{e_j(\mathcal{H})}\right) \cdot r^{e_{z(y^*)+1}(\mathcal{H}) + \cdots + e_r(\mathcal{H})} \\
&\le {Y^*_{r,k}}^{\zeta(\mathcal{H})m^2} \cdot r^{\mathrm{ex}(m,K_k) - \zeta(\mathcal{H})m^2} \le {Y^*_{r,k}}^{\xi m^2} \cdot r^{\mathrm{ex}(m,K_k) - \xi m^2}.
\end{aligned} \tag{14}$$

Table 1. $r_0^*(k)$ for $3 \leq k \leq 15$.

k	3	4	5	6	7	8	9	10	11	12	13	14	15
$r_0^*(k)$	12	74	182	346	576	882	1273	1757	2344	3041	3859	4806	5890

Finally, by (7), (14), our choice of ξ, and as $Y^*_{r,k} < r$, we have that

$$r^{(r\eta+H(r\eta))n^2} \cdot \left(\frac{Y^*_{r,k}}{r}\right)^{\xi n^2} \cdot r^{\mathrm{ex}(n,K_k)} \overset{n\gg 1;(3)}{\ll} r^{\mathrm{ex}(n,K_k)}, \tag{15}$$

a contradiction. This concludes the proof of the Claim 4.1. □

5. Concluding remarks

By using an LP-solver, we may compute the value of $r_0^*(k)$ for any fixed value of k. For $k \in \{3, \ldots, 15\}$, the values are given in Table 1. As it turns out, so far the best value of $Y^*_{r,k}$ in (13) was achieved for $y = \ell$.

The focus of our paper is on lower bounds for $r_0(k)$. For $k \geq 4$ and n large, it is easy to find upper bounds on $r_0(k)$. Consider complete s-partite graphs $G = (V, E)$, $s = s(r)$, with a balanced partition $V = V_1 \cup \cdots \cup V_s$. We may split the set of r colors into $\binom{s}{2}$ sets $S_{i,j}$, $1 \leq i < j \leq s$, in a balanced way, and consider colorings of G where edges between V_i and V_j are assigned colors in $S_{i,j}$. Clearly, any copy of a complete subgraph in G is rainbow. For sufficiently large values of r, it is easy to fix s such that G has more $\mathcal{P}_k$-free colorings than $T_{k-1}(n)$. For the case of $k = 3$, there is a similar construction where color classes are matchings, which gives more r-colorings than $T_2(n)$ for all $r \geq 27$.

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