Certified CNF Translations for Pseudo-Boolean Solving

Stephan Gocht, Ruben Martins, Jakob Nordström and **Andy Oertel**

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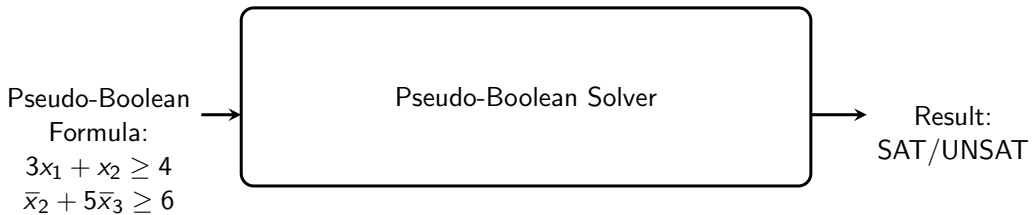
The Pseudo-Boolean (PB) Problem



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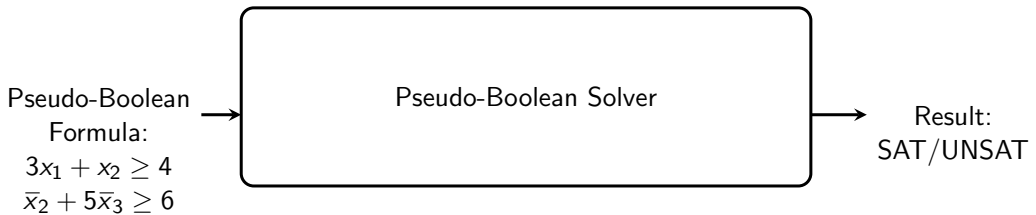

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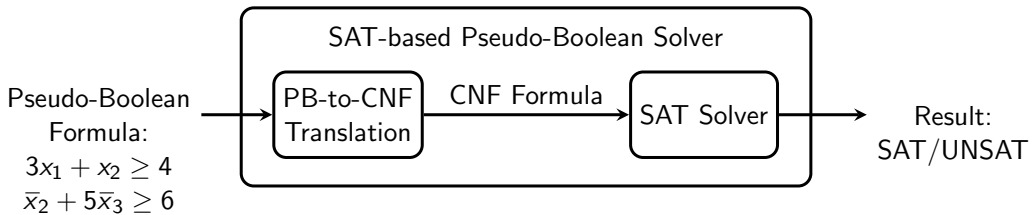

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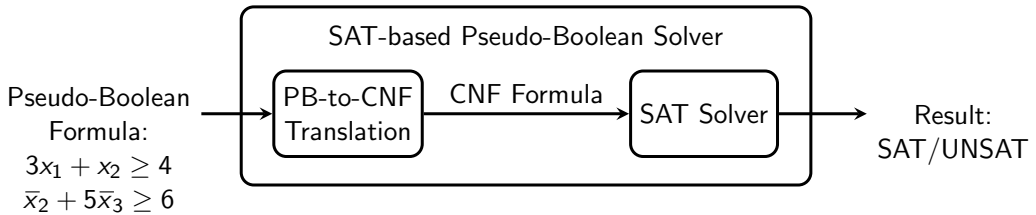

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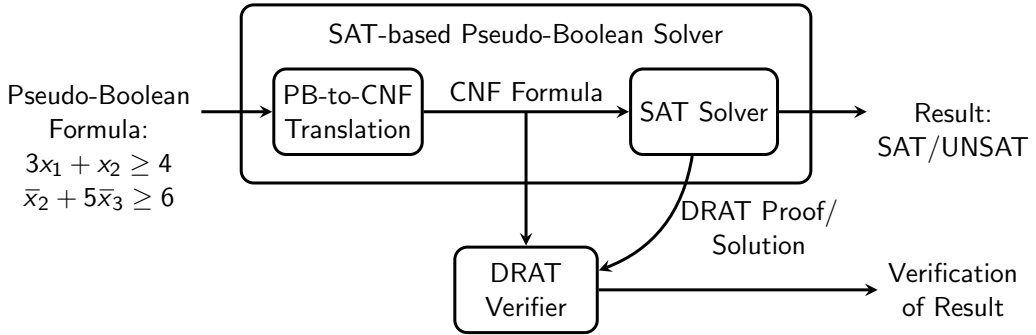

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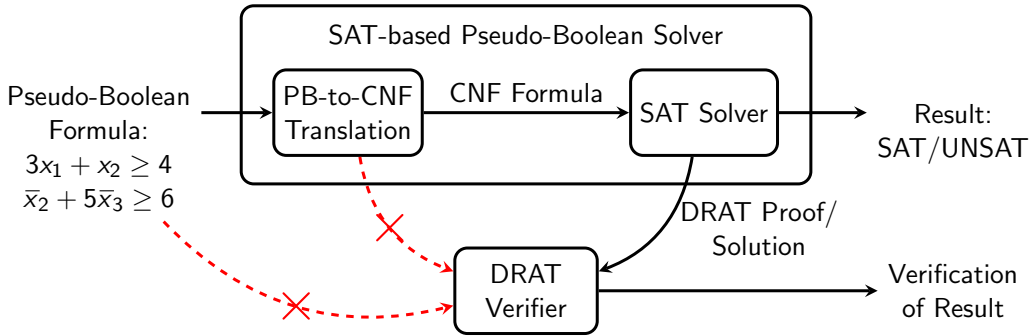

- Correctness of SAT solver solution can be certified [HHW13a, HHW13b, WHH14]
- **PB-to-CNF translation uncertified!**

Pseudo-Boolean Proof Logging

- Multi-purpose proof format
- Allows easy proof logging for
  - Reasoning with pseudo-Boolean constraints (by design)
  - SAT solving (including advanced techniques) [GN21, BGMN22]
  - Constraint Programming [EGMN20, GMN22]
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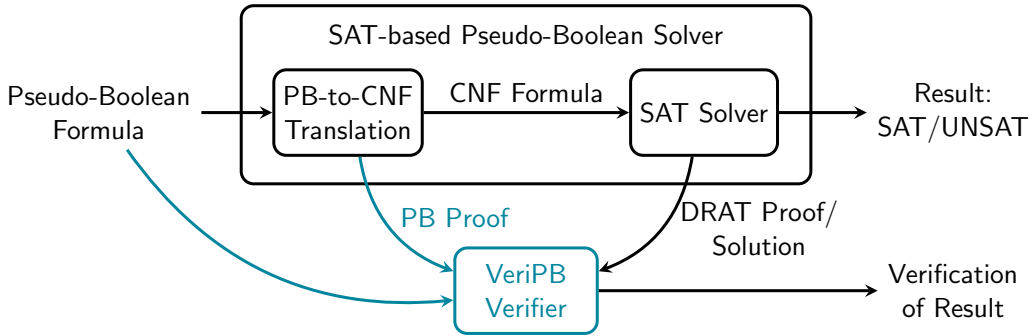
This Work

- Proof logging for translating pseudo-Boolean constraints to CNF
- General framework to certify many different encodings
- Promising foundation for proof logging MaxSAT solving and PB optimization

Workflow


SAT-based Pseudo-Boolean Solver
Pseudo-Boolean Formula
PB-to-CNF Translation
CNF Formula
SAT Solver
Result: SAT/UNSAT
PB Proof
DRAT Proof/ Solution
VeriPB Verifier
Verification of Result

Basic Notation

- Boolean variable x: with domain 0 (false) and 1 (true)
- Literal ℓ: x or negation $\overline{x} = 1 - x$
- Pseudo-Boolean (PB) constraint: integer linear inequality over literals

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- Equality constraint: syntactic sugar for 2 inequalities

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- Clause: disjunction of literals / at-least-one constraint

$$x_1 \vee \overline{x}_2 \vee \overline{x}_3 \iff x_1 + \overline{x}_2 + \overline{x}_3 \geq 1$$

Cutting Planes Proof System [CCT87]

Rules:

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$$\frac{}{\ell_i \geq 0}$$

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- Division (and rounding up)

$$\text{Divide by 2}\ \frac{2x_1 + 2\overline{x}_2 + 4x_3 \geq 5}{x_1 + \overline{x}_2 + 2x_3 \geq 3}$$

Extended Cutting Planes: Reification

Extension rule to introduce fresh variables:

- Reification (special case of redundance rule in [GN21, BGMN22])

$$a \Leftrightarrow x_1 + \overline{x}_2 + 2x_3 \geq 2 \longrightarrow \begin{array}{ll} 2\overline{a} + x_1 + \overline{x}_2 + 2x_3 \geq 2 & (a \Rightarrow x_1 + \overline{x}_2 + 2x_3 \geq 2) \\ 3a + \overline{x}_1 + x_2 + 2\overline{x}_3 \geq 3 & (a \Leftarrow x_1 + \overline{x}_2 + 2x_3 \geq 2) \end{array}$$

Translating PB to CNF: Outline

1. Construct circuit evaluating left-hand side of pseudo-Boolean constraint
2. Encode circuit to CNF using so-called Tseitin translation
3. Enforcing inequality

Translating PB to CNF: Step 1

1. **Construct circuit evaluating left-hand side of pseudo-Boolean constraint**

- Several approaches to construct logical circuit evaluating PB constraint
  - Sequential counter [Sin05], totalizer [BB03], adder network [ES06], ...

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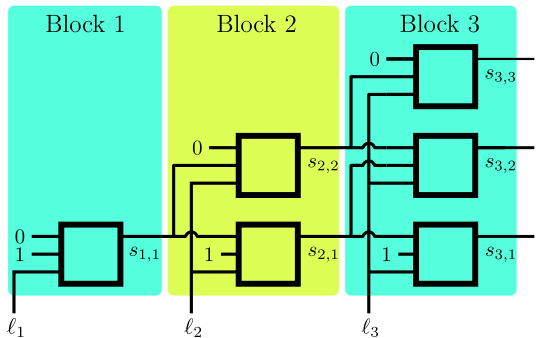
Example: $\ell_1 + \ell_2 + \ell_3 \geq 2$

Meaning of $s_{i,j}$ variable:
$s_{i,j}$ true if and only if
$\ell_1 + \ldots + \ell_i \geq j$



Translating PB to CNF: Step 2

2. **Encode circuit to CNF using so-called Tseitin translations**

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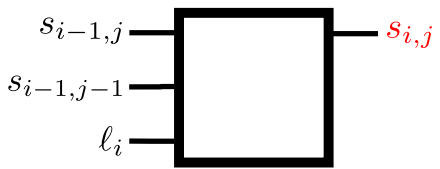
Example: Sequential counter component



Specification of $s_{i,j}$

$$s_{i,j} \leftrightarrow (\ell_i \wedge s_{i-1,j-1}) \vee s_{i-1,j}$$

Clausal encoding

$$\bar{\ell}_i \vee \bar{s}_{i-1,j-1} \vee s_{i,j}$$
$$\bar{s}_{i-1,j} \vee s_{i,j}$$
$$\ell_i \vee s_{i-1,j} \vee \bar{s}_{i,j}$$
$$s_{i-1,j-1} \vee \bar{s}_{i,j}$$

Translating PB to CNF: Step 3

3. **Enforcing inequality**

- Add clauses enforcing comparison with right-hand side

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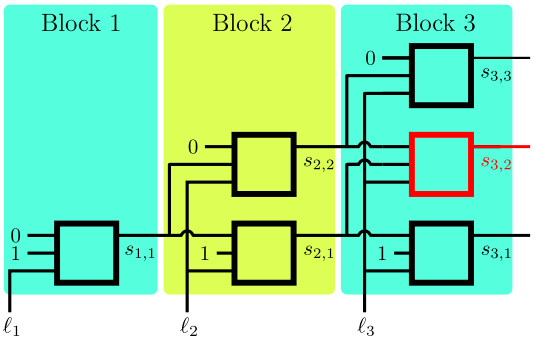
Example: $\ell_1 + \ell_2 + \ell_3 \geq 2$



At least 2 true literals if $s_{3,2}$ true.

Add unary clause

$s_{3,2}$

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This means

- Pseudo-Boolean formula satisfiable $\implies$ CNF translation satisfiable
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End-to-end verification of SAT-based pseudo-Boolean solvers!

Rest of This Presentation: How Is This Done?

We develop general framework certifying PB-to-CNF translations!

- But let us stay with our example:

Sequential counter encoding of $\ell_1 + \ell_2 + \ell_3 \geq 2$

Circuit Specification in Pseudo-Boolean Form

We want to derive

- Specification of $s_{i,j}$ variables

$$s_{i,j} \Leftrightarrow \sum_{k=1}^{i-1} s_{i-1,k} + \ell_i \geq j$$

- Ordering of $s_{i,j}$ variables

$$s_{i,j} \geq s_{i,j+1}$$

- Preservation of sum

$$\sum_{k=1}^{i} s_{i,k} = \sum_{k=1}^{i-1} s_{i-1,k} + \ell_i$$

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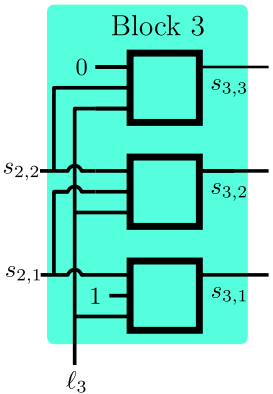

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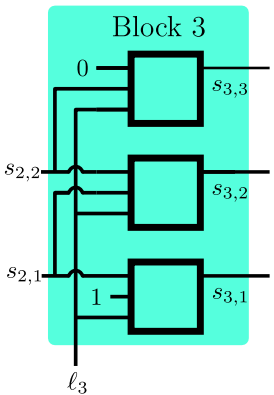

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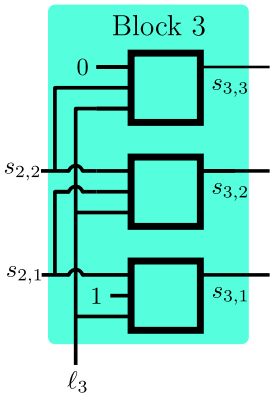

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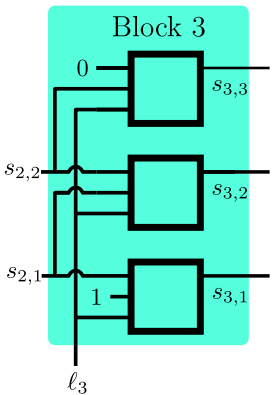

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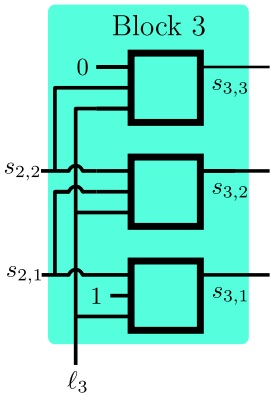

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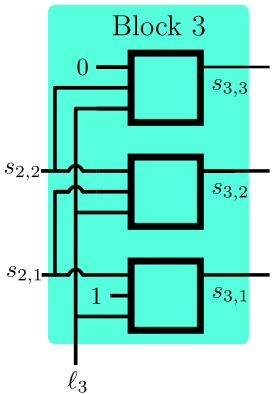

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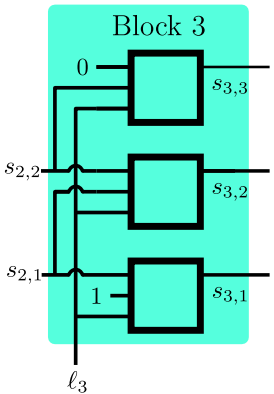

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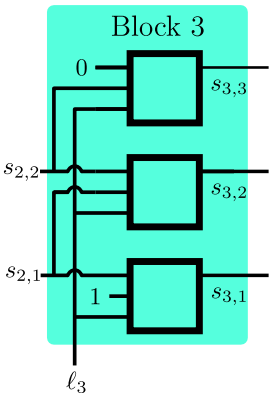

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We want to derive:

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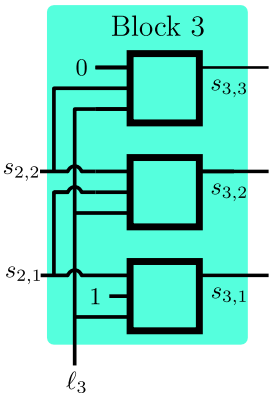

Deriving Pseudo-Boolean Specification for Block 3

$$\bar{s}_{3,1} + s_{2,1} + s_{2,2} + \ell_3 \geq 1 \qquad 3s_{3,1} + \bar{s}_{2,1} + \bar{s}_{2,2} + \bar{\ell}_3 \geq 3$$
$$2\bar{s}_{3,2} + s_{2,1} + s_{2,2} + \ell_3 \geq 2 \qquad 2s_{3,2} + \bar{s}_{2,1} + \bar{s}_{2,2} + \bar{\ell}_3 \geq 2$$
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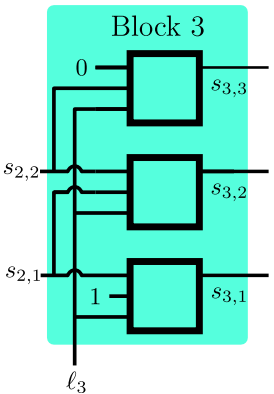

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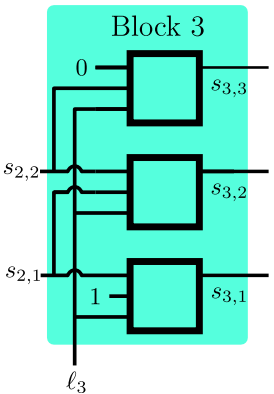

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- Can be derived using standard Cutting Planes derivations
- This is straightforward — see paper for details

Telescoping Preservation Equality

$$s_{1,1} = \ell_1$$
$$s_{2,1} + s_{2,2} = s_{1,1} + \ell_2$$
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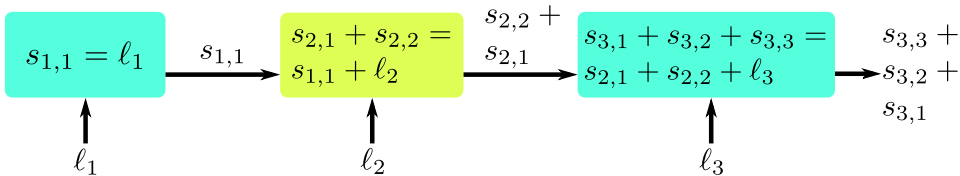

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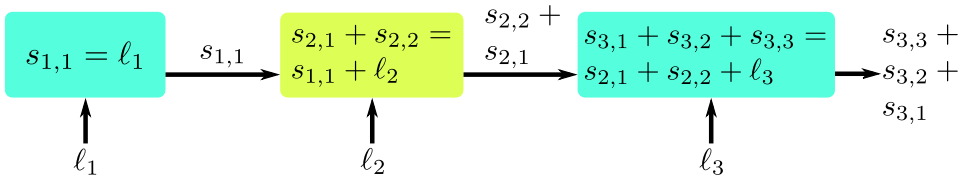

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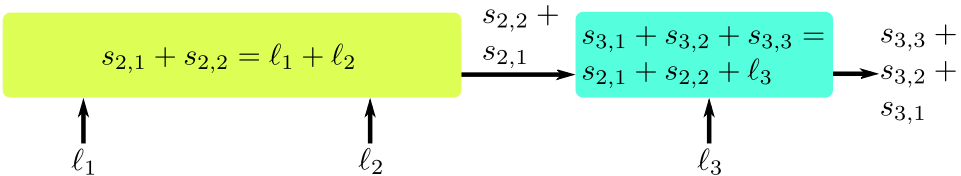

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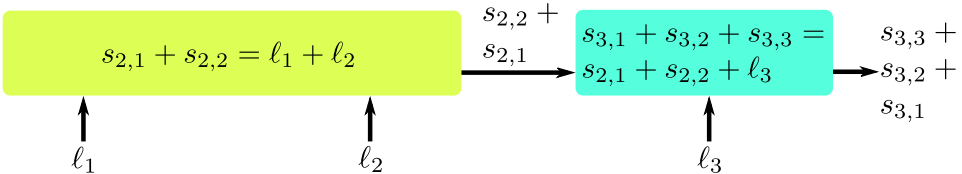

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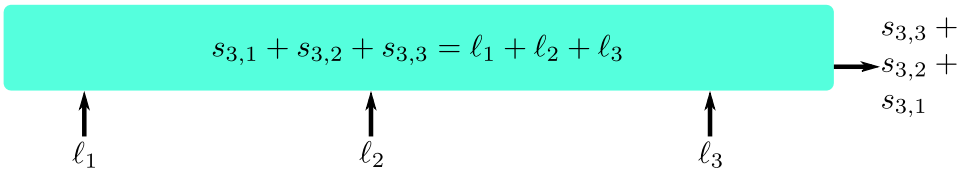

Enforcing Bound on Output Variables

$$s_{3,1} + s_{3,2} + s_{3,3} = \ell_1 + \ell_2 + \ell_3$$

We want a bound on the output variables:

$\ell_1 + \ell_2 + \ell_3 \geq 2$	pseudo-Boolean input
$s_{3,1} + s_{3,2} + s_{3,3} = \ell_1 + \ell_2 + \ell_3$	just derived

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$\Rightarrow s_{3,1} + s_{3,2} + s_{3,3} \geq 2$	telescoping again

Deriving the CNF Translation

We now have pseudo-Boolean constraints:

$$s_{1,1} = \ell_1 \qquad s_{2,1} + s_{2,2} = s_{1,1} + \ell_2 \qquad s_{3,1} + s_{3,2} + s_{3,3} = s_{2,1} + s_{2,2} + \ell_3$$
$$s_{2,1} \geq s_{2,2} \qquad s_{3,1} \geq s_{3,2} \qquad s_{3,2} \geq s_{3,3} \qquad s_{3,1} + s_{3,2} + s_{3,3} \geq 2$$

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But we want clauses:

$$\bar{\ell}_1 \vee s_{1,1}$$
$$\ell_1 \vee \bar{s}_{1,1}$$
$$\bar{\ell}_2 \vee s_{2,1}$$
$$\bar{s}_{1,1} \vee s_{2,1}$$
$$\ell_2 \vee s_{1,1} \vee \bar{s}_{2,1}$$

$$\bar{\ell}_2 \vee \bar{s}_{1,1} \vee s_{2,2}$$
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But we want clauses:

$$\begin{array}{rrrr}
\bar{\ell}_1 \vee s_{1,1} & \bar{\ell}_2 \vee \bar{s}_{1,1} \vee s_{2,2} & \ell_3 \vee s_{2,1} \vee \bar{s}_{3,1} & \bar{\ell}_3 \vee \bar{s}_{2,2} \vee s_{3,3} \\
\ell_1 \vee \bar{s}_{1,1} & \ell_2 \vee \bar{s}_{2,2} & \bar{\ell}_3 \vee \bar{s}_{2,1} \vee s_{3,2} & \ell_3 \vee \bar{s}_{3,3} \\
\bar{\ell}_2 \vee s_{2,1} & s_{1,1} \vee \bar{s}_{2,2} & \bar{s}_{2,2} \vee s_{3,2} & s_{2,2} \vee \bar{s}_{3,3} \\
\bar{s}_{1,1} \vee s_{2,1} & \bar{\ell}_3 \vee s_{3,1} & \ell_3 \vee s_{2,2} \vee \bar{s}_{3,2} & s_{3,2} \\
\ell_2 \vee s_{1,1} \vee \bar{s}_{2,1} & \bar{s}_{2,1} \vee s_{3,1} & s_{2,1} \vee \bar{s}_{3,2} &
\end{array}$$

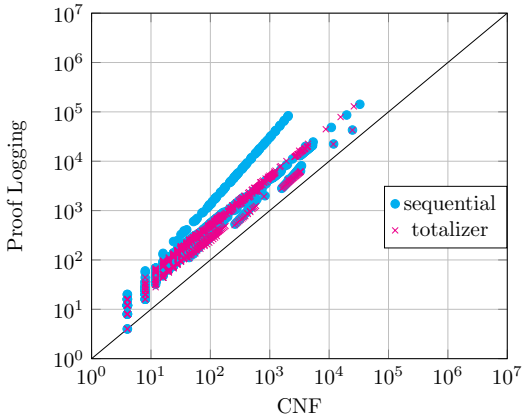
- Clauses follow from PB specification by reverse unit propagation [GN03, Van08]
- See paper for details

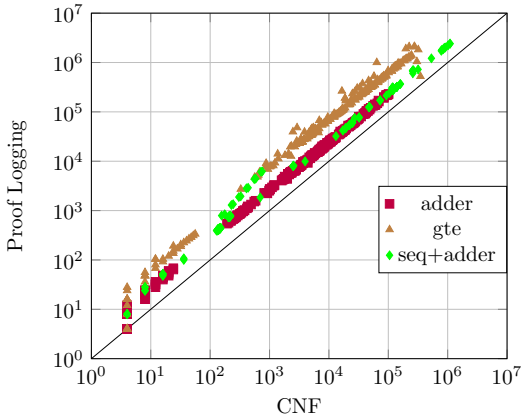
Experiments

- Implemented CNF translations with proofs of correctness[1] for
  - Sequential counter [Sin05]
  - Totalizer [BB03]
  - Generalized totalizer [JMM15]
  - Adder network [ES06]
- Proof verified by proof checker VeriPB[2]
- Benchmarks from PB 2016 Evaluation[3]
  - `SMALLINT` decision benchmarks without purely clausal formulas
  - 3 subclasses of benchmarks:
    - Only cardinality constraints (sequential counter, totalizer)
    - Only general PB constraints (generalized totalizer, adder network)
    - Mixed cardinality & general PB constraints (sequential counter + adder network)

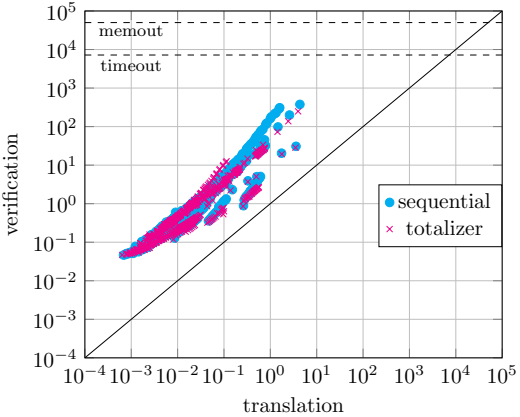
[1] `https://github.com/forge-lab/VeritasPBLib`
[2] `https://gitlab.com/MIAOresearch/VeriPB`
[3] `http://www.cril.univ-artois.fr/PB16/`

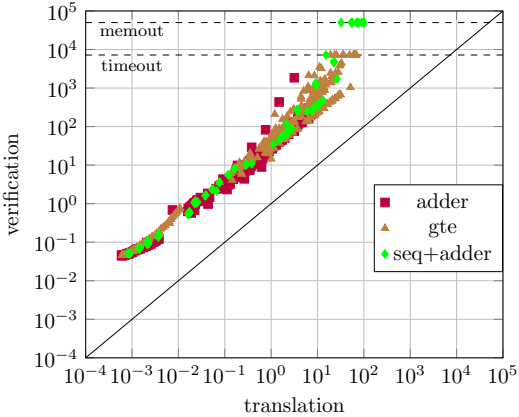
CNF Size vs Proof Size in KiB





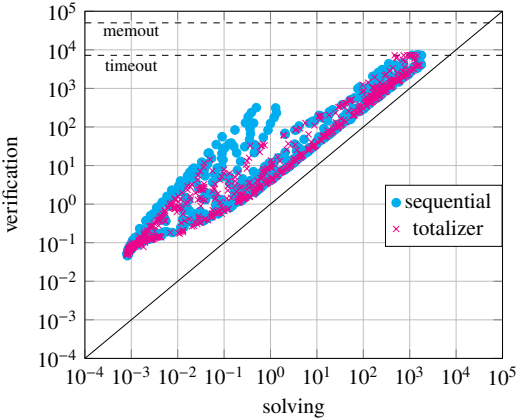
Translation Time vs Verification Time in Seconds

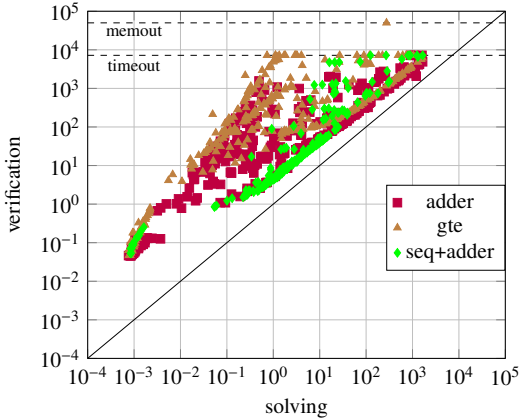




- Translation just generates clauses and proof
- Verification slower, as reasoning has to be performed

Solving Time vs Verification Time in Seconds





- Solved with fork of Kissat[4] syntactically modified to output pseudo-Boolean proofs
- Room for improvement, but this clearly shows that our approach is viable

[4] `https://gitlab.com/MIAOresearch/kissat_fork`

Future Work

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Sounds interesting? Join us! We are hiring!

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Thank you for your attention!

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