Certified Core-Guided MaxSAT Solving

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Extended Reunion: Satisfiability 2023

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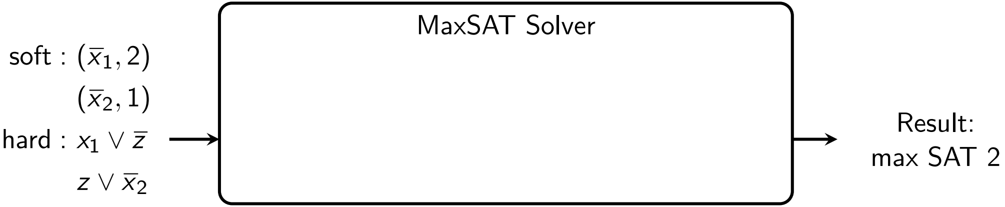
Joint work with Jeremias Berg, Bart Bogaerts, Jakob Nordström and Dieter Vandesande

Maximum Satisfiability (MaxSAT) Solving



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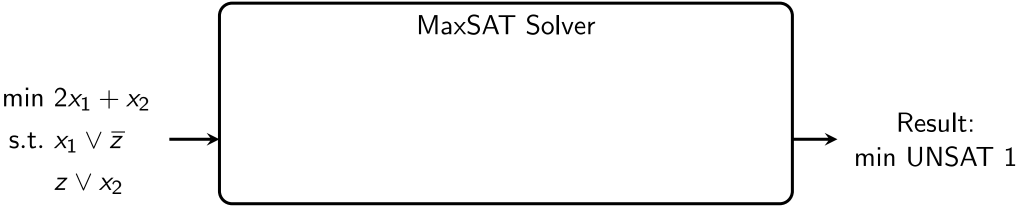

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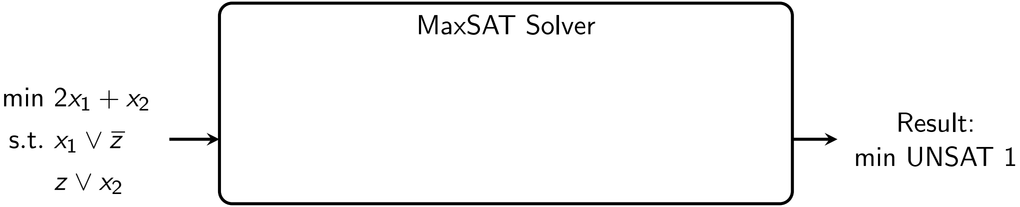
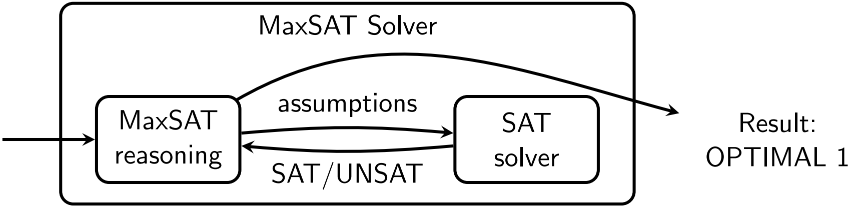


- Maximize sum of weight of satisfied soft clauses subject to satisfying hard clauses
- Alternatively: Minimize linear objective subject to satisfying a CNF formula
- Main approaches:
  - Solution-improving or linear SAT-UNSAT search [ES06, LP10, PRB18]
  - Implicit hitting set (IHS) search [DB13a, DB13b]
  - **Core-guided search** [FM06, NB14, ADR15, AG17]

Core-Guided MaxSAT Solving

$$\min\ 2x_1 + x_2 \quad \text{s.t.}\ x_1 \vee \bar{z},\quad z \vee x_2$$



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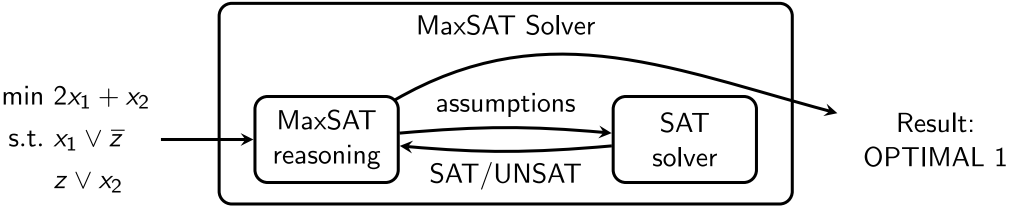
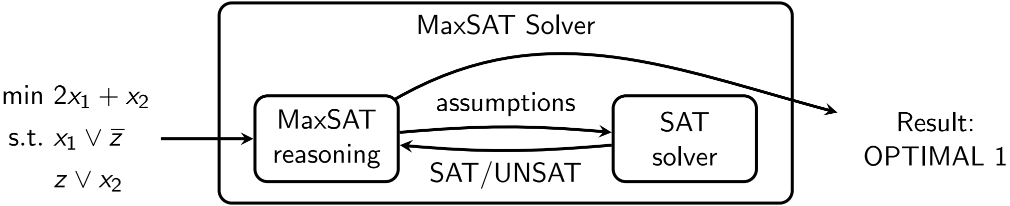


- Call SAT solver with optimistic assumptions (assignment to objective variables)
- If UNSAT, SAT solver returns core clause over assumptions explaining UNSAT
  - Reformulate objective, relax assumptions, increase lower bound
- If SAT, optimal solution found

Correctness of Core-Guided MaxSAT Solving



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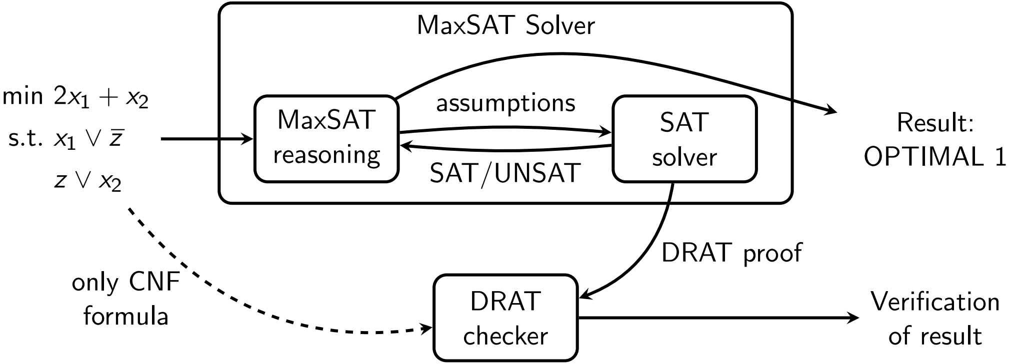

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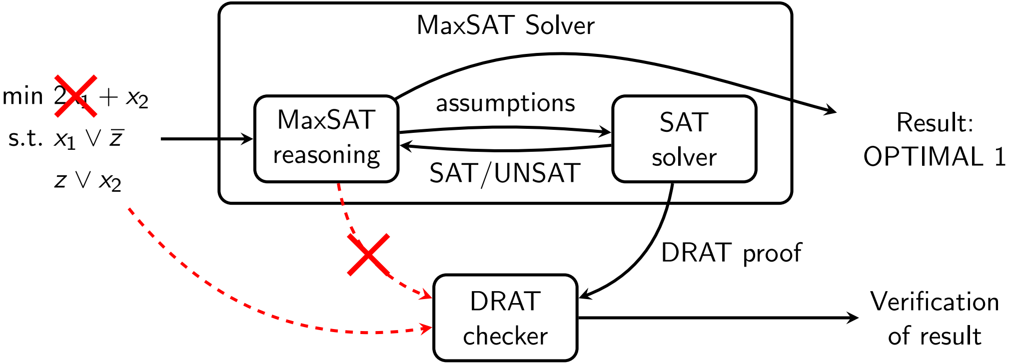

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- **Core-guided MaxSAT reasoning not certified!**

Pseudo-Boolean (PB) Proof Logging

- **Multi-purpose** proof format
- Allows easy proof logging for
  - Reasoning with pseudo-Boolean constraints (by design)
  - SAT solving (including advanced techniques) [GN21, BGMN22]
  - Constraint programming [EGMN20, GMN22]
  - Subgraph problems [GMN20, GMM+20]
  - SAT-based pseudo-Boolean solving [GMNO22]
  - Unweighted linear SAT-UNSAT search MaxSAT [VDB22]

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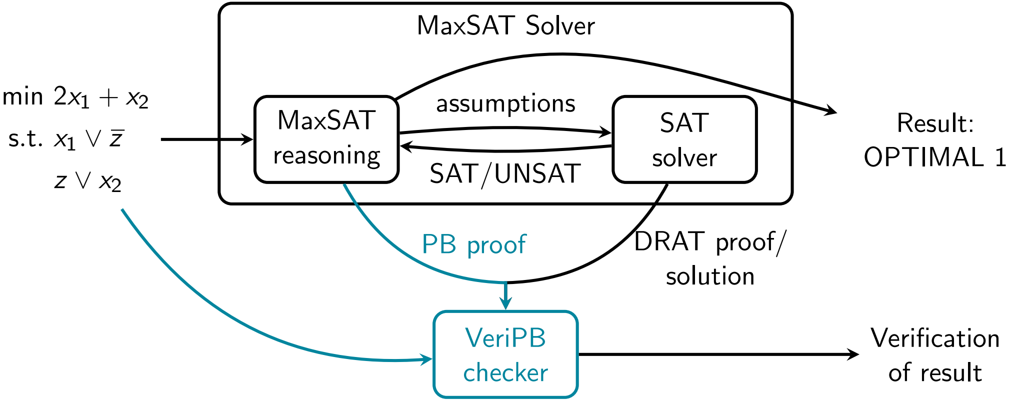
This work:

- Proof logging for state-of-the-art core-guided MaxSAT solving [IMM19, IBJ21]

Workflow



- Single PB proof that interleaves MaxSAT reasoning and SAT solver proof

Basic Notation

- Boolean variable x: with domain 0 (false) and 1 (true)
- Literal ℓ: x or negation $\overline{x} = 1 - x$
- Pseudo-Boolean (PB) constraint: integer linear inequality over literals

$$3x_1 + 2x_2 + 5\overline{x}_3 \geq 5$$

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$$3x_1 + 2x_2 + 5\overline{x}_3 = 5 \longrightarrow \begin{array}{l} 3x_1 + 2x_2 + 5\overline{x}_3 \geq 5 \\ 3x_1 + 2x_2 + 5\overline{x}_3 \leq 5 \end{array}$$

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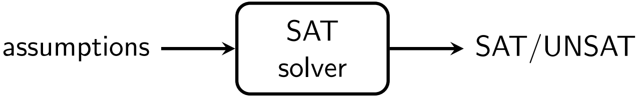
- Clause: disjunction of literals / at-least-one constraint

$$x_1 \vee \overline{x}_2 \vee \overline{x}_3 \iff x_1 + \overline{x}_2 + \overline{x}_3 \geq 1$$

Calling SAT Solver with Assumptions



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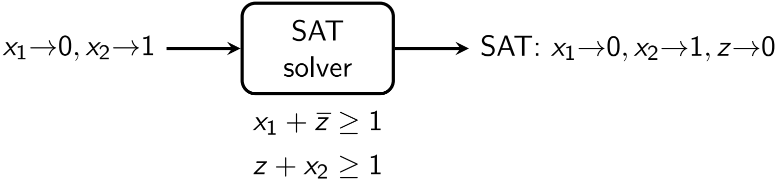

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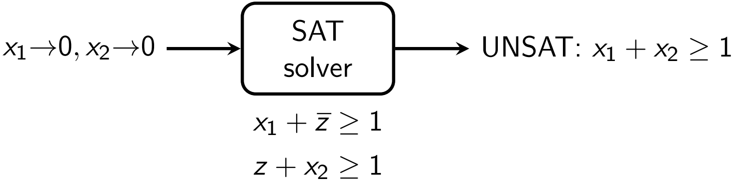

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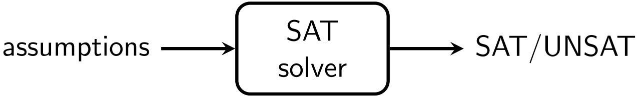

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- Supported by modern SAT solvers

OLL-Style Core-Guided MaxSAT Solving [MDM14]

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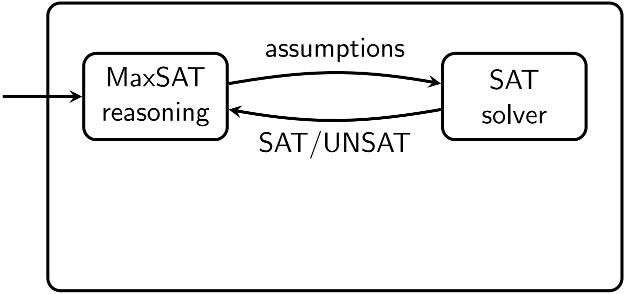

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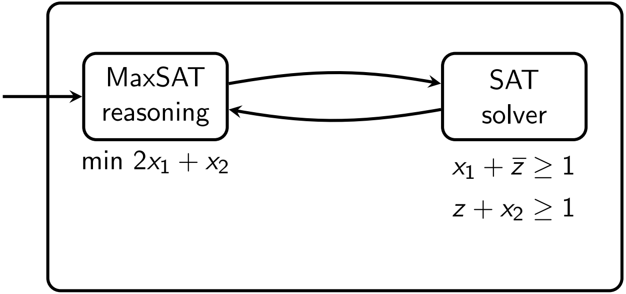

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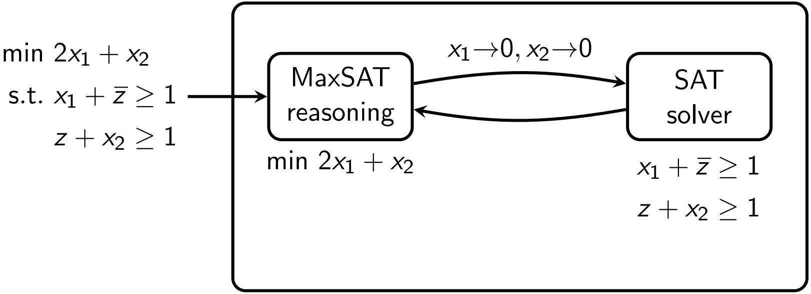

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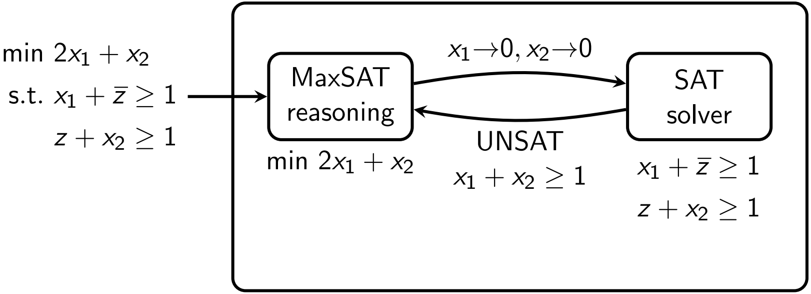

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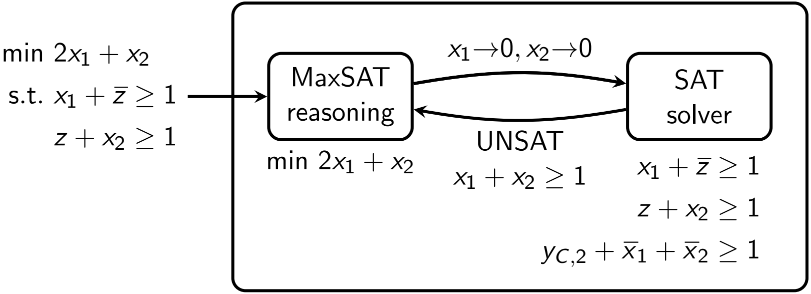

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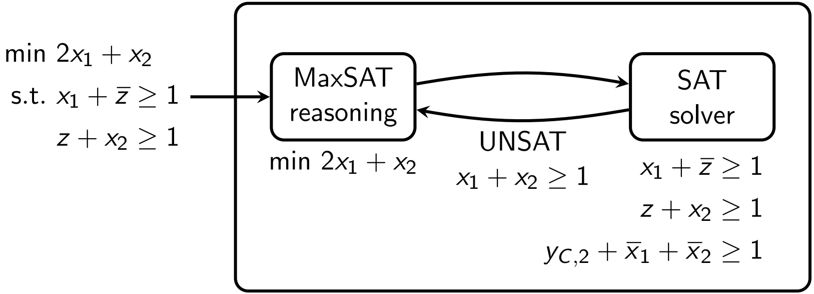

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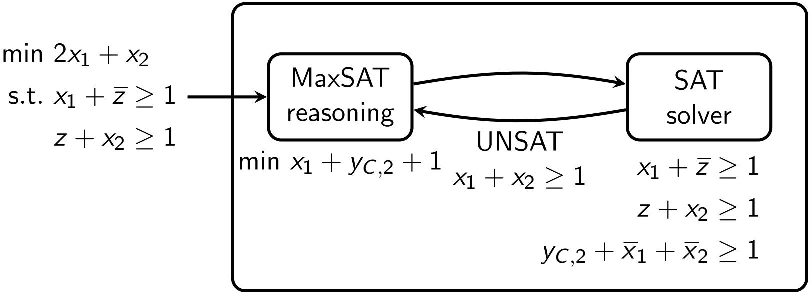

- Introduce counter variable $y_{C,2} \Leftrightarrow x_1 + x_2 \geq 2$
- Reformulate objective with $x_1 + x_2 = 1 + y_{C,2}$
- Substitute core literals in objective with counter variables

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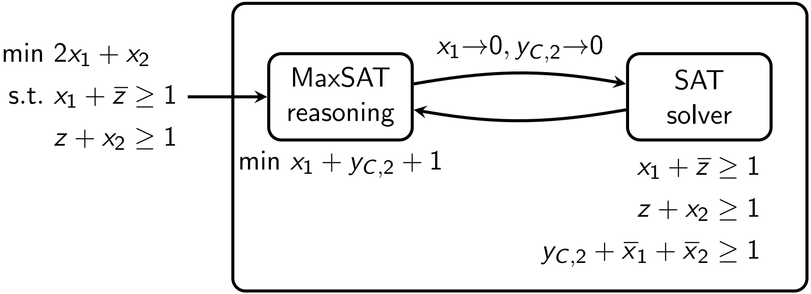

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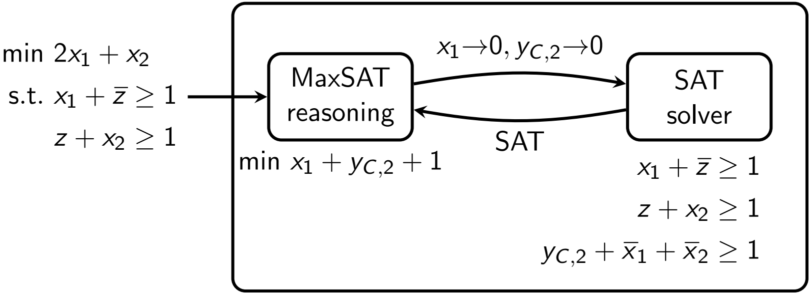

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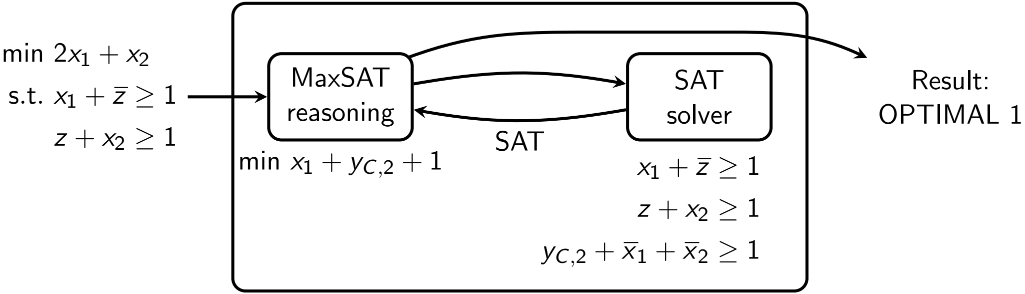

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- Optimal solution found with value 1

Our Work: Correctness of Algorithm

- State-of-the-art solvers are way more complex [IMM19, IBJ21]
- Algorithm correct in theory [AKMS12, MDM14]
- But did we implement it correctly?

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Verification of core-guided MaxSAT solving!

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- Literal axiom

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- Division (and rounding up)

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Extended Cutting Planes: Reification

- Reification (special case of redundance rule in [GN21, BGMN22])

$$a \Leftrightarrow x_1 + \overline{x}_2 + 2x_3 \geq 2 \longrightarrow \begin{array}{ll} 2\overline{a} + x_1 + \overline{x}_2 + 2x_3 \geq 2 & (a \Rightarrow x_1 + \overline{x}_2 + 2x_3 \geq 2) \\ 3a + \overline{x}_1 + x_2 + 2\overline{x}_3 \geq 3 & (a \Leftarrow x_1 + \overline{x}_2 + 2x_3 \geq 2) \end{array}$$

- Variable a was not used in proof so far

Roadmap to Certifying Core-Guided MaxSAT

1. Record solution in proof to establish upper bound UB
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formula:

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If SAT solver returns SAT:

- Solution is recorded in the proof

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Certifying Totalizers

- Counter variable $y_{C,j}$ for core C true iff $\sum_{\ell \in C} \ell \geq j$ ($y_{C,j} \Leftrightarrow \sum_{\ell \in C} \ell \geq j$)
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$$C\colon\ x_1 + x_2 + x_3 \geq 1 \implies \begin{array}{rl} 2y_{C,2} + \bar{x}_1 + \bar{x}_2 + \bar{x}_3 \geq 2 & (y_{C,2} \Leftarrow x_1 + x_2 + x_3 \geq 2) \\ 2\overline{y_{C,2}} + x_1 + x_2 + x_3 \geq 2 & (y_{C,2} \Rightarrow x_1 + x_2 + x_3 \geq 2) \\ y_{C,3} + \bar{x}_1 + \bar{x}_2 + \bar{x}_3 \geq 1 & (y_{C,2} \Leftarrow x_1 + x_2 + x_3 \geq 3) \\ 3\bar{y}_{C,3} + x_1 + x_2 + x_3 \geq 3 & (y_{C,2} \Rightarrow x_1 + x_2 + x_3 \geq 3) \end{array}$$

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- Encode constraints from PB to CNF using totalizers
- Output of totalizers are counter variables
- Prove that clausal encoding follows from PB definition as in [GMNO22, VDB22]

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- For each core derive reformulation due to this core [GMNO22]

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- Add reformulation due to each core together to get $f_{orig} = f_{reform}$

Roadmap to Certifying Core-Guided MaxSAT

1. Record solution in proof to establish upper bound UB ✔
2. Establish lower bound LB ✔
   - Prove objective function reformulation (which contains LB) ✔
     - Show that core clauses are valid ✔
     - Introduce clauses defining counter variables ✔
3. Obtain proof of optimality if $LB = UB$

Proving Optimality

- **Idea:** Finding a better solution than the optimal solution is contradicting

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- Optimal solution α recorded at least once in proof log
- Introduce objective-improving constraint that enforces $f_{orig} \leq f_{orig}(\alpha) - 1$

$$\min 2x_1 + x_2 \quad \text{and} \quad \text{solution: } x_1 \to 0,\ x_2 \to 1$$

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- The result is contradicting

Certifying Advanced Techniques

- State-of-the-art core-guided MaxSAT solvers use additional techniques
  - Intrinsic at-most-one constraints [IMM19]
  - Hardening [ABGL12]
  - Lazy counter variables [MJML14]
- Proof logging also required for these techniques

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Very convenient with pseudo-Boolean reasoning!

Experiments

- Implemented certifying version of state-of-the-art solver CGSS[1] [IBJ21]
- Proof checked with proof checker VeriPB[2]
- Benchmarks from MaxSAT Evaluation 2022[3]
  - 607 unweighted instances and 594 weighted instances

[1] https://gitlab.com/MIAOresearch/software/certified-cgss
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First result:

- Discovered bug in RC2 on which CGSS is based
  - Optimal solution correct for all instances in our set, but reasoning sometimes wrong
  - Can lead to solver claiming optimality for non-optimal solution

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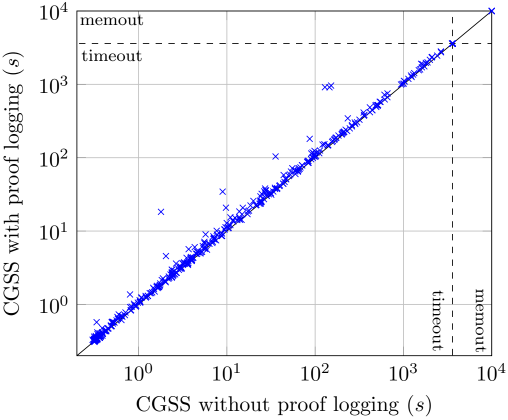
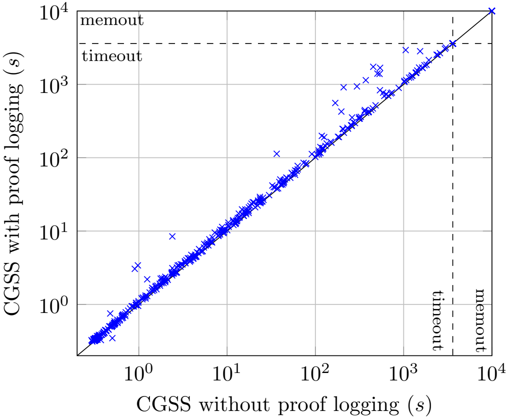
Overhead for Proof Logging in Seconds



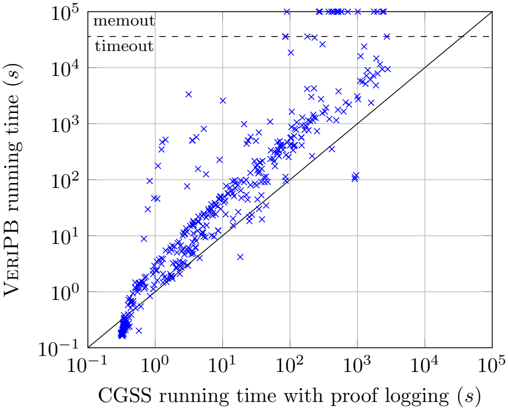
(a) Unweighted instances.



(b) Weighted instances.

- Low proof logging overhead (6.5% overhead in median)

Solving versus Checking in Seconds



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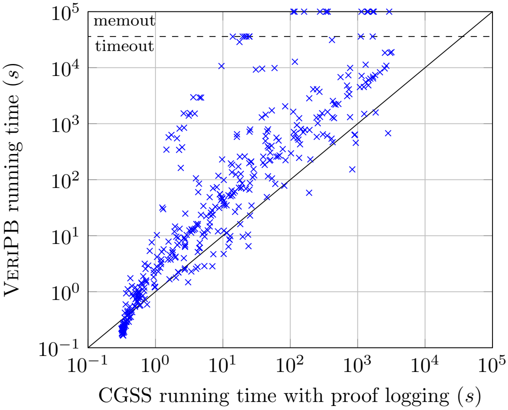

(b) Weighted instances.

- Checking time could be improved (VeriPB not optimized for SAT solver proofs)

Future Work

Further proof logging:

- State-of-the-art linear SAT-UNSAT search solver (like Pacose)
- Implicit hitting set MaxSAT solver
  - Fundamental challenge: proof logging for MIP solver
- Pseudo-Boolean optimization

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Sounds interesting? Join us! We are hiring!

Conclusion

- MaxSAT: successful optimization paradigm, but without proof logging
- DRAT not sufficient, but PB reasoning supports MaxSAT proof logging
- **This work:** Proof logging for state-of-the-art core-guided MaxSAT solving
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Pseudo-Boolean reasoning provides unified proof logging method for:

- SAT solving (including advanced techniques) [GN21, BGMN22]
- Constraint programming [EGMN20, GMN22]
- Subgraph problems [GMN20, GMM$^+$20]
- SAT-based pseudo-Boolean solving [GMNO22]
- **This work:** Core-guided MaxSAT solving

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Thank you for your attention!

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